UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3506**

ASSESSMENT PATTERN : **MATH3506A**

MODULE NAME : **Mathematical Ecology**

DATE : **13-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH3506A-001-EXAM-15

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Two interacting species with densities x, y are modelled by the system

$$\begin{aligned} \frac{dx}{dt} &= x(a - bx + cy) \\ \frac{dy}{dt} &= y(d + ex - fy) \end{aligned} \tag{1}$$

where $a, b, c, d, e, f > 0$.

   (a) Briefly discuss the model, identifying the type of species-species interactions involved. What are the carrying capacities for x and y?

   (b) Find all steady states of the system (1) and determine whether they are locally stable or unstable.

   (c) Determine $\lim_{t\to\infty} x(t), \lim_{t\to\infty} y(t)$ in the cases (i) $x(0) = 0, y(0) > 0$ and (ii) $x(0) > 0, y(0) = 0$.

   (d) Sketch the phase planes for the system (1) when (i) $ce < bf$ and (ii) $ce \geqslant bf$.

2. Consider the following model giving the size, $N_{k+1} \geqslant 0$, of a population at time $k+1$ in terms of the population size at time k:

$$N_{k+1} = rN_k \left(1 - \frac{N_k}{K}\right) = f(N_k), \quad K > 0 \text{ and } 0 < r < 4, \quad k = 0, 1, \ldots. \tag{2}$$

   (a) Determine the positive steady state for (2) and its linear stability when $1 < r < 3$.

   (b) Sketch the cobweb maps for (2) in the cases (i) $0 < r < 1$ and (ii) $1 < r < 2$.

   (c) Explain carefully how 2−cycles of (2) are related to fixed points of f and the composite function $f^2 = f \circ f$.

   (d) Prove that a stable 2−cycle appears for (2) as r increases through 3. [You may use that $r^2(1-N)(rN^2 - rN + 1) - 1 = (r - rN - 1)(1 + r - (r + r^2)N + r^2N^2)$].

   (e) Show that if $N_0 \in [0, K]$ then $N_k \in [0, K]$ for all $k = 0, 1, 2, \ldots$.

3. Two interacting populations are modelled by the planar system

$$\begin{aligned} \dot{N} &= N(a - eN - bP) \\ \dot{P} &= P(-c + dN - fP) \end{aligned} \qquad (3)$$

where $a, b, c, d, e, f > 0$ are constants.

(a) What kind of species-species interactions are involved in the model?

(b) Determine the condition(s) on the parameters a, c, d, e for there to be a unique interior steady state (N^*, P^*).

(c) By considering the function

$$V(N,P) = d\left(N - N^* \log\left(\frac{N}{N^*}\right)\right) + b\left(P - P^* \log\left(\frac{P}{P^*}\right)\right),$$

or otherwise, determine the behaviour of all interior trajectories of (3) as $t \to \infty$.

(d) Sketch the phase plane when $e = f = 0$, explaining how you arrive at your conclusions.

4. A predator-prey model has the form

$$\frac{dN}{dt} = \rho N\left(1 - \frac{N}{K}\right) - \frac{\gamma N P}{A + N}, \qquad (4)$$

where ρ, K, γ, A are all positive constants. Here N is the prey density and P is the predator density which is <u>assumed constant</u>.

(a) Show how to rewrite (4) as

$$\frac{du}{d\tau} = u(1-u) - \frac{\alpha u}{\beta + u}, \qquad (5)$$

where $\tau = \rho t$, $u = N/K$ and α, β are positive constants that you should find.

(b) Show that for $\beta > \alpha$ there are exactly two steady states for (5).

(c) Show that the necessary and sufficient condition on α, β for there to be exactly three distinct steady states is

$$1 > \beta > 2\sqrt{\alpha} - 1 \text{ and } \alpha > \beta,$$

and determine their local stability in this case.

(d) Suppose that $P = \dfrac{\rho A}{\gamma}$ and the prey rests in a positive steady state. If an influx of predator then fixes $P > \dfrac{K\rho}{4\gamma}\left(1 + \dfrac{A}{K}\right)^2$ explain carefully what happens to the prey density.

5. A population is divided into three classes. The susceptibles S are those who may contract a disease, the infectives I are those who are currently infected with the disease, and the removed class R are those who have recovered from the disease and are now immune. After a period of time, recovered individuals lose their immunity and become susceptible to infection. The disease is not fatal and births and deaths are not included in the model. The model is given by

$$\begin{aligned}\frac{dS}{dt} &= -rIS + \kappa R\\ \frac{dI}{dt} &= rIS - aI\\ \frac{dR}{dt} &= aI - \kappa R,\end{aligned}$$

where r, a, κ are positive constants. The initial conditions are $S(0) = S_0 > 0, I(0) = I_0 > 0$ and $R(0) = 0$.

(a) Briefly state the meaning of the constants r, a, κ.

(b) Show how to eliminate S and obtain a self-contained planar system for I, R only.

(c) Show that if $S_0 + I_0 > \dfrac{a}{r}$ there are two steady states.

(d) Sketch the $I - R$ phase plane for the case $S_0 + I_0 > \dfrac{a}{r}$.

(e) Prove that the infection always dies out if $S_0 + I_0 < \dfrac{a}{r}$.